12th Oral Exam of S.-T. Yau College Student Mathematics Contests 2021

Analysis and differential equations
Group Contest

1. Explain that the dual space of $L^\infty(\mathbb{R})$ is not $L^1(\mathbb{R})$.

2. Let $\varphi : [0,1] \to \mathbb{R}$ be integrable for the Lebesgue measure. Define $G : \mathbb{R} \to \mathbb{R}_+$ by
$$G(t) = \int_{[0,1]} |\varphi(x) - t| dx.$$
   (i) Show that G is continuous on $\mathbb{R}$;
   (ii) Show that G is derivable at $t \in \mathbb{R}$ if and only if
$$\lambda_1(\{x : \varphi(x) = t\}) = 0,$$
   here λ_1 denotes the Lebesgue measure on $\mathbb{R}^1$.

3. Let $B_1(0)$ be the unit ball in $\mathbb{R}^3$ centered at the origin. Assume that the function v is a smooth function defined on $\mathbb{R}^3$ with $v_r = \frac{x\cdot\nabla v}{|x|} \in L^2(B_1(0)$. Prove that
$$\begin{aligned}\int_{B_1(0)} \frac{|v(x)|^2}{|x|^2} dx &\le C(\int_{B_1(0)} |v_r|^2 dx + \int_{\partial B_1(0)} |v|^2 d\sigma) \\ &\le C_1 \int_{B_1(0)} (|v_r|^2 + |v|^2) dx,\end{aligned}$$
   where C and C_1 are some constants independent of v.

4. Prove that the life span of any solution to the following differential equation
$$\frac{dy}{dx} = x^2 + y^2$$
   is finite.